

A criterion for the existence of zero modes for the Pauli operator with fastly decaying fields

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Abstract

We consider the Pauli operator in \mathbb{R}^3 for magnetic fields in $L^{3/2}$ that decay at infinity as $|x|^{-2-\beta}$ with $\beta > 0$. In this case we are able to prove that the existence of a zero mode for this operator is equivalent to a quantity $\delta(\mathbf{B})$, defined below, being equal to zero. Complementing a result from [4], this implies that for the class of magnetic fields considered, Sobolev, Hardy and CLR inequalities hold whenever the magnetic field has no zero mode.

1 Introduction

Consider the Pauli operator $\mathbf{P}_{\mathbf{A}}$ acting on $L^2(\mathbb{R}^3, \mathbb{C}^2) \equiv \mathcal{H}$, formally defined by

$$\mathbf{P}_{\mathbf{A}} = (\mathbf{p} - \mathbf{A})^2 - \boldsymbol{\sigma} \cdot \mathbf{B}$$

where $\mathbf{B} = \text{curl } \mathbf{A}$. In appropriate units, this operator describes the kinetic energy of a non-relativistic electron in the magnetic field \mathbf{B} . We will also need the Schrödinger operator $\mathbf{S}_{\mathbf{A}} = (\mathbf{p} - \mathbf{A})^2$, which gives the kinetic energy of a spinless particle in a magnetic field. An element of the kernel of $\mathbf{P}_{\mathbf{A}}$ is called a zero mode for the corresponding Pauli operator.

The importance of zero modes for the Pauli operator was first pointed out in [9], where the authors realized that their existence would imply a critical value of the nuclear charge Z in order to have a bounded ground state energy for a one-electron atom in a magnetic field. In [12], the first examples of magnetic fields producing zero modes were given. Further examples were given in [1, 2, 6, 8]. [2] provides explicit examples of magnetic fields with an arbitrary number of zero modes while in [6] a compactly supported magnetic field having a zero mode is constructed. In [8] the authors use a geometrical approach which allows, for a certain class of magnetic fields on \mathbb{R}^3 , to relate the problem to the one on \mathbb{S}^2 , which is better understood.

All of the above papers deal with the problem of describing the kernel of the Pauli operator for fixed magnetic fields. A different point of view is adopted in [3] and [7]. In these cases the authors describe the set of magnetic fields producing zero modes, in [3] for $\mathbf{B} \in L^{3/2}$ and in [6] for continuous \mathbf{A} decaying as $o(|x|^{-1})$.

Both authors reach the conclusion that magnetic fields on \mathbb{R}^3 producing zero modes are rather *rare* which contrasts heavily with the situation in \mathbb{R}^2 .

The existence of zero modes for the Pauli operator makes it impossible to use the kinetic energy of a wave function to control its potential energy as it is done for (magnetic) Schrödinger operators by Hardy's inequality or the CLR-bound ([5, 11, 13]). However, in [4] it was shown that it is still possible to obtain this type of bounds for certain magnetic fields. Here, the goal is to give a more precise description of the class of magnetic fields for which this bound holds. In order to make this statement precise, we first need to review some results of [3, 4].

If $|\mathbf{B}| \in L^q$ for some $q \in [\frac{3}{2}, \infty]$, $\mathbf{S}_{\mathbf{A}}$ and $\mathbf{P}_{\mathbf{A}}$ have the same form domain $\mathcal{Q}(\mathbf{S}_{\mathbf{A}})$. Both operators can be defined as Friedrich's extensions of the respective quadratic forms. In addition, we will need the operator $\tilde{\mathbf{P}}_{\mathbf{A}} \equiv \mathbf{P}_{\mathbf{A}} + |\mathbf{B}|$, with the same form domain. Since $\tilde{\mathbf{P}}_{\mathbf{A}} \geq \mathbf{S}_{\mathbf{A}}$, $\ker(\tilde{\mathbf{P}}_{\mathbf{A}}) = \{0\}$, so its range is dense in \mathcal{H} . The auxiliary Hilbert space $\tilde{\mathcal{H}}$ is defined as the completion of $\mathcal{Q}(\mathbf{S}_{\mathbf{A}})$ with respect to the norm

$$\|u\|_{\tilde{\mathcal{H}}}^2 = (u, \tilde{\mathbf{P}}_{\mathbf{A}} u).$$

This space is not a subspace of \mathcal{H} . Its definition ensures $\tilde{\mathbf{P}}_{\mathbf{A}}^{-1/2}$ considered as an operator from $\text{Ran}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2})$ to $\tilde{\mathcal{H}}$ preserves norms. As previously remarked, its domain is dense in \mathcal{H} . On the other hand, $\text{Ran}(\tilde{\mathbf{P}}_{\mathbf{A}}^{-1/2}) = \mathcal{D}(\mathbf{P}_{\mathbf{A}}^{1/2}) = \mathcal{Q}(\mathbf{P}_{\mathbf{A}}) = \mathcal{Q}(\mathbf{S}_{\mathbf{A}})$, which is dense in $\tilde{\mathcal{H}}$ by construction. This means $\tilde{\mathbf{P}}_{\mathbf{A}}^{-1/2}$ can be extended to a unitary operator U from \mathcal{H} to $\tilde{\mathcal{H}}$. Multiplication by $|\mathbf{B}|^{1/2}$ is a bounded operator from $\tilde{\mathcal{H}}$ to \mathcal{H} . This allows us to define

$$\begin{aligned} S &= |\mathbf{B}|^{1/2} U : \mathcal{H} \rightarrow \mathcal{H}, \\ S &= |\mathbf{B}|^{1/2} (\mathbf{P}_{\mathbf{A}} + |\mathbf{B}|)^{-1/2} \quad \text{on } \text{Ran}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2}). \end{aligned}$$

Finally, define

$$\delta(\mathbf{B}) = \inf_{\|f\|=1, Uf \in \mathcal{H}} \|(1 - S^* S)f\|. \quad (1)$$

With these definitions, we can state the main result.

Theorem 1.1. *If $\mathbf{B} \in L^{3/2}$ is such that $\delta(\mathbf{B}) = 0$ and there exists $\beta > 0$, $C \geq 0$ and $r_0 \geq 0$ such that*

$$|\mathbf{B}|(x) \leq C|x|^{-2-\beta}$$

for all $|x| \geq r_0$, then the associated Pauli operator $\mathbf{P}_{\mathbf{A}}$ has a zero mode.

We do not know whether the condition on the decay of \mathbf{B} is optimal. In any case it can be replaced by the condition on the vector potential \mathbf{A} in hypothesis of lemma 3.2. Our method does not work without this additional decay of \mathbf{A} .

The quantity $\delta(\mathbf{B})$ was introduced in [4] where the following result was proven:

Theorem 1.2 (Balinsky, Evans, Lewis, [4]). *If $\mathbf{B} \in L^{3/2}$, then*

$$\mathbf{P}_{\mathbf{A}} \geq \delta(\mathbf{B}) \mathbf{S}_{\mathbf{A}}. \quad (2)$$

If $\delta(\mathbf{B}) > 0$, this result allows to deduce for instance a Hardy inequality for $\mathbf{P}_{\mathbf{A}}$. If the Pauli operator corresponding to the magnetic field \mathbf{B} has a zero mode, then $\delta(\mathbf{B}) = 0$. The content of theorem 1.1 is precisely the converse of this. For magnetic fields that decrease sufficiently fast at infinity, $\delta(\mathbf{B}) = 0$ implies the existence of a zero mode for the corresponding Pauli operator. Unfortunately, inequality (2) still contains the positive but unknown quantity $\delta(\mathbf{B})$.

The remainder of this paper contains the proof of theorem 1.1. The next section contains some preliminary lemmas while the third section concludes the proof.

2 Simplifying the problem

To prove theorem 1.1 we will first simplify the statement, by reducing the condition $\delta(\mathbf{B}) = 0$ to a simpler one and changing the hypothesis on the decay of \mathbf{B} into a hypothesis on \mathbf{A} . This is done in the following two lemmas.

Lemma 2.1. *If $\delta(\mathbf{B}) = 0$, then*

$$\inf_{\substack{g \in \mathcal{Q}(\mathbf{S}_{\mathbf{A}}) \\ (g, |\mathbf{B}|g) \neq 0}} \frac{(g, \mathbf{P}_{\mathbf{A}}g)}{(g, |\mathbf{B}|g)} = 0. \quad (3)$$

Proof. First, observe that if

$$\inf_{\|f\|=1, Uf \in \mathcal{H}} \|(1 - S^*S)f\| = 0,$$

then

$$\sup_{\|f\|=1, Uf \in \mathcal{H}} \|Sf\| = 1.$$

To see this, first notice that for any $f \in \mathcal{H}$, $\|Sf\| \leq \|f\|$, so the *sup* in the above expression is at most 1. Now if f_n is a minimizing sequence for the first problem,

$$(1 - S^*S)f_n \rightarrow 0 \text{ in } L^2$$

so in particular

$$(f_n, (1 - S^*S)f_n) \rightarrow 0.$$

This means $\|Sf_n\|^2 = (f_n, S^*Sf_n) \rightarrow 1$.

Since the range of $\tilde{\mathbf{P}}_{\mathbf{A}}$ is dense in \mathcal{H} and S is bounded, nothing is lost by restricting the sup to functions $f \in \text{Ran}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2})$. For these functions the condition $Uf \in \mathcal{H}$ is trivially satisfied. The problem can then be rewritten in terms of $g = Uf$:

$$\begin{aligned} 1 &= \sup_{\|f\|=1, Uf \in \mathcal{H}} \|Sf\| = \sup_{f \in \text{Ran}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2}) \setminus \{0\}} \frac{\|Sf\|}{\|f\|} \\ &= \sup_{g \in \mathcal{D}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2}) \setminus \{0\}} \frac{\| |\mathbf{B}|^{1/2} g \|}{\| \tilde{\mathbf{P}}_{\mathbf{A}}^{1/2} g \|} \end{aligned}$$

The result is obtained by expanding $\|\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2}g\|^2 = (g, \mathbf{P}_{\mathbf{A}}g) + (g, |\mathbf{B}|g)$ and using $\mathcal{D}(\tilde{\mathbf{P}}_{\mathbf{A}}^{1/2}) = \mathcal{Q}(\mathbf{S}_{\mathbf{A}})$:

$$1 = \sup_{\|f\|=1, Uf \in \mathcal{H}} \|Sf\|^2 = \sup_{g \in \mathcal{Q}(\mathbf{S}_{\mathbf{A}}) \setminus \{0\}} \left(\frac{(g, \mathbf{P}_{\mathbf{A}}g)}{(g, |\mathbf{B}|g)} + 1 \right)^{-1},$$

which is only possible if

$$\inf_{\substack{g \in \mathcal{Q}(\mathbf{S}_{\mathbf{A}}) \\ (g, |\mathbf{B}|g) \neq 0}} \frac{(g, \mathbf{P}_{\mathbf{A}}g)}{(g, |\mathbf{B}|g)} = 0. \quad \square$$

Then, we show that the imposed decay of \mathbf{B} implies a good decay of \mathbf{A} if we fix the gauge

$$\frac{1}{4\pi} \mathbf{A}(x) \equiv \int \frac{x-y}{|x-y|^3} \times \mathbf{B}(y) dy. \quad (4)$$

Note that \mathbf{A} as defined above is in L^3 by the weak Young inequality.

Lemma 2.2. *If $\mathbf{B} \in L^{3/2}$ is such that there exists $\beta > 0$, $C_B \geq 0$ and $r_0 \geq 0$ such that*

$$|\mathbf{B}|(x) \leq C_B |x|^{-2-\beta}$$

for all $|x| \geq r_0$, then there exist $r_1 \geq r_0$ and C_A such that

$$|\mathbf{A}|(x) \equiv 4\pi \left| \int \frac{x-y}{|x-y|^3} \times \mathbf{B}(y) dy \right| \leq C_A |x|^{-1-\alpha}$$

for $\alpha = \min(1/2, \beta/2)$ and all $|x| \geq r_1$

Proof. Take $r_1 = \max((2r_0)^2, 1)$. Take any x such that $|x| \geq r_1$ and define $r_x = |x|^{1/2}/2 \geq r_0$. Split the domain of integration in the definition of \mathbf{A} in two parts and apply Hölder's inequality to the first part to obtain

$$\begin{aligned} |\mathbf{A}|(x) &\leq 4\pi \int_{B_{r_x}} |\mathbf{B}(y)| |x-y|^{-2} dy + 4\pi \int_{\overline{B_{r_x}}} |\mathbf{B}(y)| |x-y|^{-2} dy \\ &\leq 4\pi \|\mathbf{B}\|_{3/2} \left(\int_{B_{r_x}} |x-y|^{-6} dy \right)^{1/3} + 4\pi C_B \int_{\overline{B_{r_x}}} |y|^{-2-\beta} |x-y|^{-2} dy \end{aligned}$$

The integrand in the first term is bounded, so

$$\begin{aligned} \int_{B_{r_x}} |x-y|^{-6} dy &\leq \frac{4\pi}{3} r_x^3 (|x| - r_x)^{-6} \\ &\leq \frac{2^5 \pi}{3} |x|^{-9/2} \end{aligned}$$

The second integral requires some more care:

$$\begin{aligned} \int_{\overline{B_{r_x}}} |y|^{-2-\beta} |x-y|^{-2} dy &= 4\pi \int_{r_x}^{\infty} r^{-\beta} dr \int_{-1}^1 dt (|x|^2 + r^2 - 2r|x|t)^{-1} \\ &= 2\pi |x|^{-1} \int_{r_x}^{\infty} r^{-\beta-1} \ln \left(\frac{|x|+r}{||x|-r|} \right) dr \\ &= 2\pi |x|^{-1-\beta} \int_{r_x/|x|}^{\infty} t^{-\beta-1} \ln \left(\frac{1+t}{|1-t|} \right) dt. \end{aligned}$$

This last integral is finite since for large t , the integrand is bounded by a constant times $t^{-\beta-1}$, while for t close to 1 it diverges only as a logarithm. Separating the range of integration in $r_x/x \leq t \leq 1/2$ and $t > 1/2$ we note that the first part gives a contribution that behaves as $C_1(r_x/|x|)^{-\beta}$ while the contribution of the second part can be bounded by a constant. This means

$$\begin{aligned} \int_{B_{r_x}} |y|^{-2-\beta} |x-y|^{-2} dy &\leq |x|^{-1-\beta} \left(C_1 \left(\frac{r_x}{|x|} \right)^{-\beta} + C_2 \right) \\ &\leq C_1 2^\beta |x|^{-1-\beta/2} + C_2 |x|^{-1-\beta}. \end{aligned}$$

We conclude

$$|\mathbf{A}|(x) \leq C_A(|x|^{-1-1/2} + |x|^{-1-\beta/2}) \leq 2C_A|x|^{-1-\alpha}. \quad \square$$

3 Compactness and Integrability

Now we use a compactness-argument to find a candidate zero mode if the infimum in equation (3) equals zero.

Lemma 3.1. *If $\mathbf{B} \in L^{3/2}$, and $\delta(\mathbf{B}) = 0$ then there exist $g \in W_{\text{loc}}^{1,2} \cap L^6$ such that*

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A})g = 0$$

in the particular gauge for \mathbf{A} defined in (4).

Proof. Take (g_n) a minimizing sequence for the problem (3) with $(g_n, |\mathbf{B}|g_n) = 1$. Then $(g_n, \mathbf{P}_\mathbf{A}g_n)$ is bounded, which implies by the diamagnetic inequality that (pg_n) is bounded in L^2 so (g_n) is bounded in L^6 . By the Banach-Alaoglu theorem, this guarantees the existence of a subsequence such that pg_n converges weakly in L^2 to some pg and $g_n \rightharpoonup g$ weakly in L^6 . Since $|\mathbf{B}| \in L^{3/2}$, this implies $(g, |\mathbf{B}|g) = 1$, so $g \neq 0$. In addition, since $\mathbf{A} \in L^3$, $(\mathbf{A}g_n)$ is bounded in L^2 so we can assume $\mathbf{A}g_n \rightharpoonup \mathbf{A}g$ weakly in L^2 . Using the fact that L^p -norms are weakly lower-semi-continuous, we obtain $\|\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A})g\|_2 = 0$. \square

To conclude the proof of theorem 1.1 we only need to show that this candidate zero mode is in L^2 . This is achieved by using the decay of \mathbf{A} given by lemma 2.2 in a bootstrap argument. The procedure is not that straightforward since the decay of $\mathbf{A}g$ and the Pauli equation imply only a decay of $\boldsymbol{\sigma} \cdot \mathbf{p}g$, which does not directly imply the decay of $\mathbf{p}g$.

Lemma 3.2. *If there exist $\alpha > 0$ and $r_1 > 0$ such that $|\mathbf{A}|(x) < C_A|x|^{-1-\alpha}$ for all $x \in \mathbb{R}^3$ with $|x| \geq r_1$ and $g \in W_{\text{loc}}^{1,2} \cap L^p$, with $p \geq 2$, is such that*

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A})g = 0,$$

then $g \in L^2$.

In order to prove this lemma, one more technical lemma will be necessary. Its proof can be found in the appendix. The inner product in $L^2(\mathbb{S}^2, \mathbb{C}^2)$ will be denoted by $\langle \cdot, \cdot \rangle$. When f and g are defined on all of \mathbb{R}^3 , we will abuse notation and write $\langle f, g \rangle(r) \equiv \langle f(r\boldsymbol{\omega}), g(r\boldsymbol{\omega}) \rangle$. We will also use the notation $\langle f \rangle(r) = \langle f, f \rangle^{1/2}(r)$.

Lemma 3.3. *If $f \in W_{loc}^{1,2}(\mathbb{R}^3)$ then $\langle f \rangle \in W^{1,2}([a, b])$ for all $b > a > 0$, and its weak derivative equals*

$$h(r) = \begin{cases} \langle f \rangle^{-1}(r) \Re \langle f, \partial_r f \rangle & \text{if } \langle f \rangle(r) > 0 \\ 0 & \text{else.} \end{cases}$$

In particular $\langle f \rangle$ is continuous except maybe at 0.

Proof of lemma 3.2. Define

$$K = -1 - \boldsymbol{\sigma} \cdot \mathbf{L}.$$

which can be considered as a self-adjoint operator on $L^2(\mathbb{S}^2, \mathbb{C}^2)$ with eigenvalues $\pm 1, \pm 2, \dots$ (see for instance [10], section 1.5). Write $g = g_+ + g_-$ where $\langle g_+, K g_+ \rangle > 0$ and $\langle g_-, K g_- \rangle < 0$. If $g \in L^p(\mathbb{R}^3)$, there exists $C > 0$ such that

$$\int_{\mathbb{S}^2} |g|^p(r\omega) d\omega \leq C r^{-3}.$$

By Jensen's inequality, this implies

$$\begin{aligned} C r^{-3} &\geq \int_{\mathbb{S}^2} |g|^p(r\omega) d\omega \geq (4\pi)^{1-p/2} \left(\int_{\mathbb{S}^2} |g|^2(r\omega) d\omega \right)^{p/2} \\ &= (4\pi)^{1-p/2} (\langle g_+, g_+ \rangle + \langle g_-, g_- \rangle)^{p/2}, \end{aligned}$$

so both $\langle g_+ \rangle(r)$ and $\langle g_- \rangle(r)$ decay as $C r^{-3/p}$.

At first, we will prove the theorem in the case that g_+ and g_- are C^2 -functions. The Pauli operator can be written conveniently as

$$\boldsymbol{\sigma} \cdot \mathbf{p} = (\boldsymbol{\sigma} \cdot \hat{\mathbf{x}})^2 \boldsymbol{\sigma} \cdot \mathbf{p} = -i \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \left(\partial_r + \frac{K+1}{r} \right),$$

where the operator inside the parenthesis commutes with K . This allows to rewrite the equation for g as

$$\partial_r g + \frac{K+1}{r} g = i \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \boldsymbol{\sigma} \cdot \mathbf{A} g.$$

For shortness, define $\sigma_A = \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \boldsymbol{\sigma} \cdot \mathbf{A}$. The only property of this matrix needed is $\|\sigma_A(r\omega)\| \leq C_A r^{-1-\alpha}$ when $r \geq r_1$. Taking the \mathbb{C}^2 product with g_+ and g_- and integrating over \mathbb{S}^2 , we obtain

$$\begin{aligned} \langle g_+, \partial_r g_+ \rangle(r) &= -\frac{1}{r} \langle g_+, (K+1)g_+ \rangle(r) + i \langle g_+, \sigma_A(g_+ + g_-) \rangle \\ \langle g_-, \partial_r g_- \rangle(r) &= -\frac{1}{r} \langle g_-, (K+1)g_- \rangle(r) + i \langle g_-, \sigma_A(g_+ + g_-) \rangle. \end{aligned} \quad (5)$$

By taking the real part of these equations, we obtain a differential equation for $\langle g_+ \rangle$ and $\langle g_- \rangle$:

$$\begin{aligned} \frac{d}{dr} \langle g_+ \rangle^2 &= -2 \frac{1}{r} \langle g_+, (K+1)g_+ \rangle(r) - 2 \Im \langle g_+, \sigma_A(g_+ + g_-) \rangle \\ \frac{d}{dr} \langle g_- \rangle^2 &= -2 \frac{1}{r} \langle g_-, (K+1)g_- \rangle(r) - 2 \Im \langle g_-, \sigma_A(g_+ + g_-) \rangle. \end{aligned}$$

Defining $\bar{g}_+ = g_+ r^2$ we get the system of equations

$$\begin{aligned}\frac{d}{dr}\langle \bar{g}_+ \rangle^2 &= -2\frac{1}{r}\langle \bar{g}_+, (K-1)\bar{g}_+ \rangle(r) - 2\Im\langle \bar{g}_+, \sigma_A(\bar{g}_+ r^2 g_-) \rangle \\ \frac{d}{dr}\langle g_- \rangle^2 &= -2\frac{1}{r}\langle g_-, (K+1)g_- \rangle(r) - 2\Im\langle g_-, \sigma_A(r^{-2}\bar{g}_+ + g_-) \rangle.\end{aligned}$$

Fix $r \geq r_1$. We now use a bootstrap argument to obtain $\langle g_\pm \rangle(r) \leq Cr^{-2}$. As remarked previously $\langle g_- \rangle^2(r) \leq Cr^{-\epsilon}$ and $\langle \bar{g}_+ \rangle^2(r) \leq Cr^{4-\epsilon}$ with $\epsilon = 3/p$. We will see the equations imply $\langle g_- \rangle^2(r) \leq C'r^{-\epsilon-\alpha}$ and $\langle \bar{g}_+ \rangle^2(r) \leq C'r^{4-\epsilon_1}$ where $\epsilon_1 = \min(\epsilon + \alpha, 4)$.

For \bar{g}_+ , we can use $\langle \bar{g}_+, K\bar{g}_+ \rangle \geq \langle \bar{g}_+ \rangle^2$ in order to obtain

$$\begin{aligned}\langle \bar{g}_+ \rangle^2(r) &= \int_{r_1}^r -2s^{-1}\langle \bar{g}_+, (K-1)\bar{g}_+ \rangle(s) - 2\Im\langle \bar{g}_+, \sigma_A(\bar{g}_+ + s^2 g_-) \rangle ds + C_1 \\ &\leq 2 \int_{r_1}^r |\langle \bar{g}_+, \sigma_A \bar{g}_+ \rangle(s)| + s^2 |\langle \bar{g}_+, \sigma_A g_- \rangle(s)| ds + C_1 \\ &\leq 4CC_A \int_{r_1}^r s^{4-\epsilon-1-\alpha} + C_1 \\ &= \frac{4CC_A}{4-\epsilon-\alpha}(r^{4-\epsilon-\alpha} - 1) + C_1.\end{aligned}\tag{6}$$

For g_- , we can use the fact $\langle g_- \rangle$ tends to zero as $r \rightarrow \infty$ and $\langle g_-, Kg_- \rangle \leq -\langle g_- \rangle^2$ to write

$$\begin{aligned}\langle g_- \rangle^2(r) &= \int_r^\infty +2s^{-1}\langle g_-, (K+1)g_- \rangle(s) + 2\Im\langle g_-, \sigma_A(s^{-2}\bar{g}_+ + g_-) \rangle ds \quad (7) \\ &\leq \int_r^\infty 2|\langle g_-, \sigma_A(s^{-2}\bar{g}_+ + g_-) \rangle| ds \\ &\leq 2CC_A \int_r^\infty 2s^{-\epsilon-1-\alpha} ds \\ &= \frac{2CC_A}{\epsilon+\alpha} r^{-\epsilon-\alpha}.\end{aligned}$$

By iterating this procedure a finite number of times we reach the conclusion $\langle g_+ \rangle(r) \leq Cr^{-2}$ and $\langle g_- \rangle(r) \leq Cr^{-2}$, so $g \in L^2(\mathbb{R}^3)$. This concludes the proof of the lemma when g_+ and g_- are C^2 -functions.

In the general case, g has a decomposition in a series of spherical spinors (see for example [10], section 1.5) where the coefficients are functions of r belonging to $W_{loc}^{1,2}(\mathbb{R}_+, r^2 dr)$. By taking the projections on the positive and negative eigenspaces of K and using dominated convergence, we conclude g_+ and g_- are in $W_{loc}^{1,2}(\mathbb{R}^3)$. Thus, by Fubini's theorem, g_\pm , and $\partial_r g_\pm$ are in $L^2(\mathbb{S}^2(r))$ for almost every $r > 0$. This justifies the integration over \mathbb{S}^2 used to obtain (5).

By lemma 3.3, $\langle g_+ \rangle$ and $\langle g_- \rangle$ are in $W^{1,2}([a, b])$ for any $b > a > 0$ and thus continuous. The use of the fundamental theorem of calculus in (6) can be justified by applying it to a sequence of C^∞ -functions converging to $\langle g_+ \rangle$ pointwise and in $W^{1,2}([r_1, r])$. In the same way we can obtain $\langle g_- \rangle^2(r) = -\int_r^{r_2} \frac{d}{dr}\langle g_- \rangle^2(r) dr + \langle g_- \rangle^2(r_2)$ for any $r_2 > r > r_1$. Since $\frac{d}{dr}\langle g_- \rangle^2(r)$ is in $L^1([r, +\infty))$, we can let $r_2 \rightarrow \infty$ in order to obtain (7). \square

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Appendix A: proof of lemma 3.3

By Fubini’s theorem, f and $\partial_r f$ are in $L^2(\mathbb{S}^2(r))$ for almost every $r > 0$ and $\langle f \rangle, \langle \partial_r f \rangle$ are in $L^2_{loc}(\mathbb{R}_+, r^2 dr)$. Fix $b > a > 0$ and define the annulus $A = \{x \in \mathbb{R}^3 | a \leq |x| \leq b\}$.

Fix $\epsilon > 0$. As a first step, we will prove $f_\epsilon \equiv (\langle f \rangle^2 + \epsilon)^{1/2} \in W^{1,2}([a, b])$. Define $h_\epsilon = \langle f_\epsilon \rangle^{-1} \Re \langle f, \partial_r f \rangle$. By the Cauchy-Schwarz inequality h_ϵ is in $L^2([a, b])$. It remains to check whether h_ϵ is the distributional derivative of f_ϵ .

To this end, take a sequence $(f_n) \subset C^1(A, \mathbb{C}^2)$ approaching f in $W^{1,2}(A)$ and pointwise almost everywhere in A . This means $\langle f_n \rangle \rightarrow \langle f \rangle$ in $L^2([a, b])$ so by extracting a subsequence we may assume $\langle f_n \rangle(r) \rightarrow \langle f \rangle(r)$ for almost every $r \in [a, b]$. Define $h_n \equiv \partial_r(\langle f_n \rangle^2 + \epsilon)^{1/2}$. We have $h_n = (\langle f_n \rangle^2 + \epsilon)^{-1/2} \Re \langle f_n, \partial_r f_n \rangle(r)$. In order to conclude, we should prove that, for any test function $\phi \in C_0^\infty([a, b])$,

$$\int_a^b \phi(r) h_n(r) dr \rightarrow \int_a^b \phi(r) h_\epsilon(r) dr \quad \text{as } n \rightarrow \infty.$$

To achieve this, fix $\phi \in C_0^\infty([a, b])$ and define $\Phi_n = (\langle f_n \rangle^2 + \epsilon)^{-1/2} \phi f_n$ and $\Phi_\epsilon = f_\epsilon^{-1} \phi f$. $\Phi_n(x)$ converges to $\Phi_\epsilon(x)$ when $x \in A$ is such that $\langle f_n \rangle(|x|)$ converges to $\langle f \rangle(|x|)$ and $f_n(x) \rightarrow f(x)$, which holds for almost every x in A . Since (Φ_n) is bounded in $L^2(A)$, by dominated convergence $\Phi_n \rightarrow \Phi_\epsilon$ in $L^2(A)$. This allows us to obtain

$$\begin{aligned} \int_a^b |\phi(h_n - h_\epsilon)| &\leq \int_a^b |\langle \Phi_n, \partial_r f_n \rangle - \langle \Phi_\epsilon, \partial_r f \rangle| \\ &\leq \int_a^b |\langle \Phi_n - \Phi_\epsilon, \partial_r f_n \rangle| + \int_a^b |\langle \Phi_\epsilon, \partial_r f_n - \partial_r f \rangle| \\ &\leq a^{-2} \|\Phi_n - \Phi_\epsilon\|_{2,A} \|\partial_r f_n\|_{2,A} + a^{-2} \|\Phi_\epsilon\|_{2,A} \|\partial_r f_n - \partial_r f\|_{2,A}. \end{aligned}$$

In the last line, we used $1 \leq a^{-2} r^2$ in the domain of integration to transform the integral over an interval in an integral over A . Since f_n tends to f in $W^{1,2}(A)$, the second term tends to zero and the second factor of the first term is bounded. As previously remarked, $\Phi_n - \Phi_\epsilon$ tends to zero in $L^2(A)$ so the first term goes to zero too. This means $f_\epsilon \in W^{1,2}([a, b])$ and its distributional derivative equals h_ϵ .

Now, we can let ϵ tend to zero. Then $f_\epsilon \rightarrow \langle f \rangle$ and $h_\epsilon(r) \rightarrow h(r)$ in $L^2([a, b])$. We conclude $\langle f \rangle \in W^{1,2}([a, b])$ and $h = \frac{d}{dr} \langle f \rangle$. \square

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